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An inequality for the spectra of nonnegative matrices ☆

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ABSTRACT

Let A be a $n \times n$ entrywise nonnegative matrix and let $s_k := trace(A^k)$, $k = 1, 2, 3$. It is shown that if $n > 1$ then $n^2 s_3 - 3ns_1s_2 + 2s_1^3 + \frac{n-2}{\sqrt{n-1}}(ns_2 - s_1^2)^{\frac{3}{2}}$ is nonnegative. The result is used to show that if $(\lambda_1, \lambda_2, \overline{\lambda_2}, \lambda_4, \ldots, \lambda_n)$ is the spectrum of a nonnegative matrix where λ_2 is nonreal and $\lambda_1 = max(|\lambda_j|, j = 1, \ldots, n)$ then $(\lambda_1 + t, \lambda_2 + t, \overline{\lambda_2} + t, \lambda_4, \ldots, \lambda_n)$ need not be realizable for all $t > 0$ even when $Re(\lambda_2) \geqslant 0$.

1. Introduction

If $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a list of complex numbers then define

$$s_k := \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k \quad \text{for } k = 1, 2, \ldots \tag{1}$$

The JLL inequalities obtained by Loewy and London [1] and independently by Johnson [2], state that if σ is the spectrum of a nonnegative matrix, then

$$n^{m-1} s_{km} \geqslant s_k^m \quad \text{for all positive integers } k, m. \tag{2}$$

These inequalities have played a fundamental role in the work on the nonnegative inverse eigenvalue problem (NIEP), which asks for necessary and sufficient conditions on a list σ of complex numbers in order that σ be the spectrum of a nonnegative matrix. When such a matrix exists we say that σ is *realizable*. In this paper we present a new inequality connecting the power sums s_k. The case $n = 4$ first appeared in the solution of the NIEP for $n = 4$ in Meehan's thesis [8].

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2. Main theorem

In order to prove the main theorem we will need the following elementary result.

Lemma 1. *Suppose $u > v \geqslant w > 0$ are real numbers with $u + v + w = \alpha$, $u^2 + v^2 + w^2 = \beta$ and $u^3 + v^3 + w^3 = \gamma$. Then we can find $0 < w' < w \leqslant v \leqslant v' \leqslant u' \leqslant u$, with $u' + v' + w' = \alpha$, $u'^2 + v'^2 + w'^2 = \beta$ and $u'^3 + v'^3 + w'^3 = \gamma' < \gamma$.*

Proof. Let

$$\begin{aligned} f(x) &= (x-u)(x-v)(x-w) \\ &= x^3 + p_1x^2 + p_2x + p_3 \end{aligned}$$

and let $t > 0$. Then, from the Newton identities, the roots u', v', w' of $f(x) + t = 0$ satisfy $u' + v' + w' = \alpha$, $u'^2 + v'^2 + w'^2 = \beta$ and $u'^3 + v'^3 + w'^3 = \gamma - 3t < \gamma$. The discriminant of $f(x)$ is positive if $v \neq w$. Thus for sufficiently small t, the discriminant of $f(x) + t$ is positive if $v \neq w$, and therefore u', v', w' are real. Also, by the continuity of the discriminant, w', u', $v' > 0$ since, $w > 0$.
If $v = w$, the discriminant (with respect to x) of $f(x) + t$ is

$$-27t^3 + (-4w^3 + 4u^3 + 12uw^2 - 12u^2w)t.$$

Then, observe that the coefficient of t in the discriminant (with respect to x) of $f(x)+t$ is $4(u-v)^3 > 0$. So, for small $t > 0$, $f(x) + t = 0$ has real roots and the roots must be positive for all sufficiently small $t > 0$ (see Fig. 1). Also $f(w) + t = t > 0$ implies $w' < w$, while $f(v) + t > 0$ and $f(u) + t > 0$, imply $v \leqslant v' \leqslant u' \leqslant u$. This completes the proof. □

Remark 2. Note that the lemma may fail if $u = v > w$.

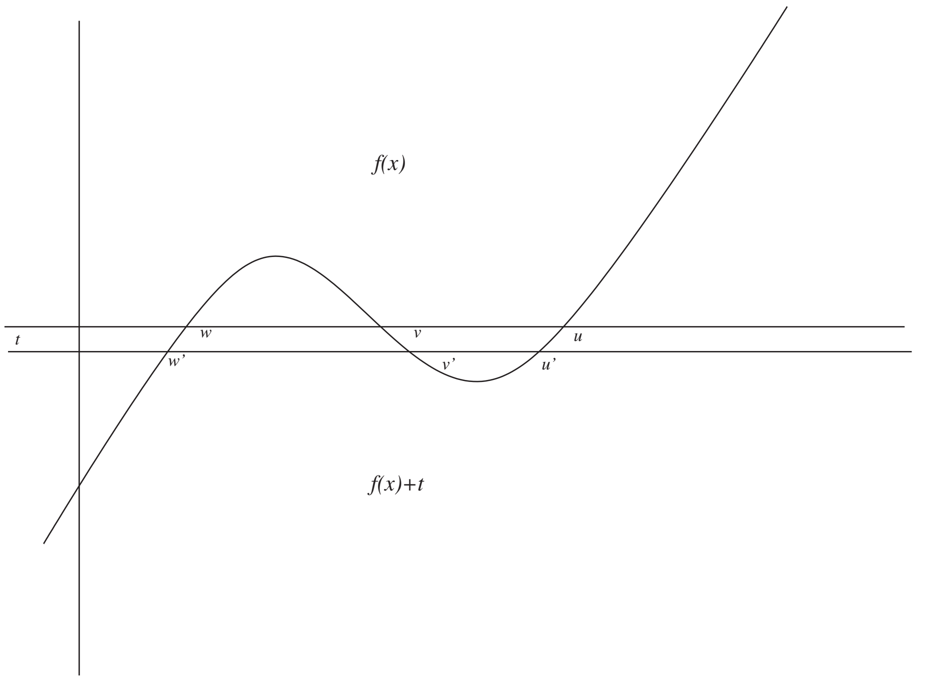

Figure 1.

Theorem 3. *Let $n > 1$ and A a nonnegative $n \times n$ matrix. Then*

$$\Phi := n^2 s_3 - 3ns_1 s_2 + 2s_1^3 + \frac{n-2}{\sqrt{n-1}}(ns_2 - s_1^2)^{\frac{3}{2}} \geqslant 0, \tag{3}$$

where $s_k = trace(A^k)$, $k = 1, 2, 3$.

Proof. Note that by (2) for $m = 2$, $k = 1$ the term $ns_2 - s_1^2 \geqslant 0$, so Φ is a real number. Also note that for $n = 2$ we get equality. Thus we can assume that $n > 2$ in proving the theorem. In proving the result we will use an idea used in proving the JLL inequalities. We write $A = D + C$ where D is a diagonal matrix and C is a matrix with zeros on the diagonal. We denote the trace of A by trA. Observe that

$$\begin{aligned} s_1 &= trD, \\ s_2 &= trD^2 + trC^2, \\ s_3 &= trD^3 + 3trDC^2 + trC^3. \end{aligned}$$

Let $C = (c_{ij})$ and note that the diagonal entries of C^2 are $z_1, z_2, \ldots, z_n$ where

$$z_i = \sum_{j \neq i} c_{ij} c_{ji} \ (i = 1, 2, \ldots, n).$$

In these terms

$$\begin{aligned} \Phi &= n^2(trD^3 + 3trDC^2 + trC^3) - 3ntrD(trD^2 + trC^2) + 2(trD)^3 \\ &\quad + \frac{n-2}{\sqrt{n-1}}(n(trD^2 + trC^2) - (trD)^2)^{\frac{3}{2}}. \end{aligned}$$

Note that $\Phi \geqslant \tilde{\Phi}$, where $\tilde{\Phi}$ is obtained from Φ by deleting the term $n^2 trC^3$.

Write $D = diag(x_1, x_2, \ldots, x_n)$ and put $z = z_1 + z_2 + \cdots + z_n = trC^2$. Then $\tilde{\Phi}$ becomes

$$\Omega := n^2 S_3 + 3n^2 \sum_{i=1}^{n} x_i z_i - 3nS_1 S_2 - 3nS_1 z + 2S_1^3 + \frac{n-2}{\sqrt{n-1}}(nS_2 + nz - S_1^2)^{\frac{3}{2}},$$

where $S_k = x_1^k + x_2^k + \cdots + x_n^k$ $(k = 1, 2, 3)$.

To prove the theorem it suffices to show that $\Omega \geqslant 0$. We consider Ω as a function of the nonnegative real variables $x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n$, where the z_i are constrained to be expressible in the form $\sum_{j \neq i} e_{ij} e_{ji}$, for some $e_{ij} \geqslant 0$. In particular, this implies that $z_i \leqslant \sum_{j \neq i} z_j$ $(i = 1, 2, \ldots, n)$.

We consider the absolute minimum of Ω subject to these constraints and $\sum_{i=1}^n x_i = X \geqslant 0$ and $\sum_{i=1}^n z_i = z \geqslant 0$ being fixed. This minimum exists since the domain defined by these inequalities is compact. At the absolute minimum, without loss of generality, we assume that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$.

Observe that the numbers z_i appear separately only in the term $3n^2 \sum z_i x_i$ and so, in minimizing Ω, the z_i should be arranged so that

$$z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n$$

by the rearrangement inequality. Furthermore, since $z = z_1 + z_2 + \cdots + z_n$ is fixed, we should arrange that the z_i are chosen so that z_n is greatest possible, then z_{n-1} is chosen greatest possible subject to this and so on. The constraint on the z_i implies that in minimizing Ω we can take $z_{n-1} = z_n = \frac{z}{2}$, thus ensuring that the contribution of $3n^2 \sum z_i x_i$ to Ω is minimized. (Note that if $x_{n-1} = 0$ then this contribution is zero.)

Next, observe that in Ω, S_3 only occurs in the term $n^2 S_3$ and so, to minimize Ω, the x_i should be chosen so that S_3 is minimal while fixing $S_1 = X$ and S_2.

Assume that the x_i, z_i are chosen, as described above, to minimize Ω. If there are indices $j_1 > j_2 > j_3$ with $x_{j_1} > x_{j_2} \geqslant x_{j_3} > 0$, where $j_1, j_2, j_3 \in \{1, 2, \ldots, n-1\}$, then, using the lemma, we can replace $x_{j_1}, x_{j_2}, x_{j_3}$ by positive real numbers $x'_{j_1}, x'_{j_2}, x'_{j_3}$ with $x'_{j_3} < x_{j_3}$ while preserving S_1 and S_2 and decreasing S_3. Note also this replacement does not increase the term $\frac{z}{2}(x_{n-1} + x_n)$, since $x'_{n-1} \leqslant x_{n-1}$ by the lemma. So Ω is decreased, contrary to our hypothesis. Note that if we were to apply the lemma to all x_i, $i = 1, 2, \ldots, n$ then, the term $\frac{z}{2}(x_{n-1} + x_n)$ is increased since $u' < u$ implies $v' + w' > v + w$, hence we omit x_n when applying the lemma.

So, there exists an integer m with $0 \leqslant m \leqslant n$ such that either

$$(1)\; x_1 = x_2 = \cdots = x_{n-m} = x > 0, \quad x_j = 0 \quad \text{for } j > n-m,$$

or

$$(2)\; x_1 = x_2 = \cdots = x_{n-m-1} = x > x_{n-m} = y > 0, \quad x_j = 0 \quad \text{for } j > n-m.$$

Suppose first that $m \geqslant 2$, i.e. at least two of the x_j are zero. Notice that since $m \geqslant 2$ the term $3n^2 \sum z_i x_i$ in Ω is zero.

In case (1),

$$\begin{aligned}
\Omega &= n^2 S_3 - 3nS_1S_2 - 3nS_1 z + 2S_1^3 + \frac{n-2}{\sqrt{n-1}}(nS_2 - S_1^2 + nz)^{3/2} \\
&= n^2(n-m)x^3 - 3n(n-m)^2x^3 - 3n(n-m)xz + 2(n-m)^3x^3 \\
&\quad + \frac{n-2}{\sqrt{n-1}}\left[n(n-m)x^2 - (n-m)^2x^2 + nz\right]^{3/2} \\
&= -m(n-m)(n-2m)x^3 - 3n(n-m)xz + \frac{n-2}{\sqrt{n-1}}\left[m(n-m)x^2 + nz\right]^{3/2}.
\end{aligned}$$

Note that when $z = 0$, $\Omega \geqslant 0$ if

$$m^3(n-2)^2(n-m)^3x^6 \geqslant m^2(n-1)(n-m)^2(n-2m)^2x^6,$$

which holds if

$$(n-2)^2m(n-m) - (n-1)(n-2m)^2 = n^2(m-1)(n-m-1) \geqslant 0,$$

which holds for $n \geqslant m+1$.

Note that if $m = n$, then all the x_i are zero and so $\Omega = 0$ when $z = 0$. Thus $\Omega \geqslant 0$ when $z = 0$.

Next we consider the behavior of Ω as z increases from 0. Fixing x and taking the derivative with respect to z gives

$$\partial\Omega/\partial z = -3n(n-m)x + \frac{3n(n-2)}{2\sqrt{n-1}}(m(n-m)x^2 + nz)^{1/2}.$$

So $\partial\Omega/\partial z$ increases with z and

$$\partial\Omega/\partial z > -3n(n-m)x + \frac{3n(n-2)}{2\sqrt{n-1}}(m(n-m))^{1/2}x$$

(since $z > 0$ and $n > 2$).

Hence $\partial\Omega/\partial z > 0$ if $(n-2)^2m \geqslant 4(n-1)(n-m)$, that is, if $n(nm-4n+4) \geqslant 0$. So $\partial\Omega/\partial z > 0$ for $z > 0$ if $m \geqslant 4$. This proves $\Omega \geqslant 0$ for all $z \geqslant 0$ if $m \geqslant 4$.

Next suppose $m < 4$. Now $\partial\Omega/\partial z = 0$ for $z = z_0$ where

$$\begin{aligned}
z_0 &= \left((2(n-m)x)^2(n-1)/(n-2)^2 - m(n-m)x^2\right)\Big/\, n \\
&= \frac{(m-n)(nm-4n+4)x^2}{(n-2)^2}.
\end{aligned}$$

Note that $z_0 > 0$ for $m \in \{2, 3\}$. This is true for $m = 2$ since $m - n < 0$, and $z_0 > 0$ for $m = 3$, since then $n \geqslant 5$ and the numerator of z_0 is positive. Now

$$\Omega(z_0) = \frac{n^2(m-n)(4n-4-2nm+m^2)x^3}{(n-2)^2}.$$

For $m = 3$, the value of $\Omega(z_0)$ is

$$\frac{(2n-5)n^2x^3(n-3)}{(n-2)^2} \geqslant 0, \text{ for } n \geqslant 3.$$

Finally for $m = 2$, $\Omega(z_0) = 0$.

Putting these facts together we have that

(i) $\partial\Omega/\partial z$ is increasing
(ii) $\partial\Omega/\partial z = 0$ for a positive solution $z = z_0$.

This implies

(iii) $\partial\Omega/\partial z < 0$ for $z < z_0$ and $\partial\Omega/\partial z > 0$ for $z > z_0$, and hence
(iv) there is a positive minimum value at $\Omega(z_0)$.

Thus we have shown that $\Omega \geqslant 0$ if $m \geqslant 2$ and $x_1 = x_2 = \cdots = x_{n-m}$.

Next suppose that (2) occurs, and again $m \geqslant 2$. First suppose $z = 0$ and write Ω as $A + B$ where

$$A = n^2S_3 - 3nS_1S_2 + 2S_1^3 \text{ and}$$
$$B = \frac{n-2}{\sqrt{n-1}}(nS_2 - S_1^2)^{\frac{3}{2}}.$$

Here

$$S_j = (n-m-1)x^j + y^j, \quad j = 1, 2, 3, \text{ where } y = x_{n-m}.$$

Then $\Omega = A + B \geqslant 0$ if $B^2 - A^2 \geqslant 0$, since $B \geqslant 0$. Since $n \geqslant m + 2$, we let $n = v + m + 2$. Setting $E = (B^2 - A^2)$ and substituting $n = v + m + 2$, we get

$$E = a_1v^2 + a_2v + a_3,$$

where

$$\begin{aligned} a_1 &= (m+1)^2x^4 - 6y(m+1)x^3 + 3y^2(m+4)x^2 - 10y^3x + 3y^4, \\ a_2 &= (m+1)^2x^4 - 6y(m+1)x^3 + 3y^2(m^2+2m+4)x^2 \\ &\quad -2y^3(6m-5)x + 3y^4(2m+1), \\ a_3 &= my^2\left(3(m+1)x^2 - 2y(m+5)x + 3(m+1)y^2\right). \end{aligned}$$

Now,

$$\begin{aligned} a_1 &= ((m+1)x^2 - 3xy)^2 + 3y^2(m+1)x^2 - 10y^3x + 3y^4 \geqslant 0 \\ &\quad \text{since } 3(m+1)x^2 - 10yx + 3y^2 = 3(y - 5x/3)^2 + 3(m - 16/9)x^2 \geqslant 0. \end{aligned}$$

Next,

$$a_2 = ((m+1)x^2 - 3xy)^2 + 3y^2(m+1)^2x^2 - 2y^3(6m-5)x + 3y^4(2m+1) \geqslant 0$$

if $3(m+1)^2x^2 - 2y(6m-5)x + 3y^2(2m+1) \geqslant 0$.

Considering this as a quadratic in x (with leading coefficient positive since $3(m+1)^2 > 0$) we find the discriminant of this quadratic to be

$$4(6m-5)^2y^2 - 4(3(m+1)^2 3(2m+1)y^2),$$

which is negative if

$$(6m-5)^2 < 9(m+1)^2(2m+1)$$

and this holds if $m > 0.1634$. Hence $a_2 \geqslant 0$ for all integers $m \geqslant 2$.

Finally, $a_3 = my^2\left(3(m+1)(x - y\frac{(m+5)}{3(m+1)})^2 + \delta\right)$ where $\delta \geqslant 0$ if $m \geqslant 1$ and thus $a_3 \geqslant 0$ for all $m \geqslant 2$. Hence we have that $E \geqslant 0$. So $\Omega = A + B \geqslant 0$ for all $m \geqslant 2$ in case (2) with $z = 0$. Now we deal with case (2) when $z > 0$. Thus $\Omega = A + B$ where

$$\begin{aligned} A &= n^2S_3 - 3nS_1S_2 - 3nS_1z + 2S_1^3 \text{ and} \\ B &= \frac{n-2}{\sqrt{n-1}}(nS_2 + nz - S_1^2)^{\frac{3}{2}}. \end{aligned}$$

Note that B is real since $nS_2 \geqslant S_1^2$ and $z \geqslant 0$. Thus in this case

$$\begin{aligned} \Omega =\ & n^2\left((n-m-1)x^3 + y^3\right) - 3n\left(((n-m-1)x + y)\left((n-m-1)x^2 + y^2\right)\right) \\ & -3n\left((n-m-1)x + y\right)z + 2\left((n-m-1)x + y\right)^3 \\ & + \frac{n-2}{\sqrt{n-1}}\left(n\left((n-m-1)x^2 + y^2\right) + nz - ((n-m-1)x + y)^2\right)^{\frac{3}{2}}. \end{aligned}$$

Fixing x and y and taking the derivative with respect to z, gives

$$\begin{aligned} \partial\Omega/\partial z =\ & -3n\left((n-m-1)x + y\right) \\ & + \frac{3n(n-2)}{2\sqrt{n-1}}\left((m+1)(n-m-1)x^2 - 2(n-m-1)xy + (n-1)y^2 + nz\right)^{\frac{1}{2}}. \end{aligned}$$

So $\partial\Omega/\partial z$ increases with z.

We will show that $\partial\Omega/\partial z > 0$. Let

$$\begin{aligned} u =& 3n\left((n-m-1)x + y\right) \text{ and} \\ v =& \frac{3n(n-2)}{2\sqrt{n-1}}\left((m+1)(n-m-1)x^2 - 2(n-m-1)xy + (n-1)y^2 + nz\right)^{\frac{1}{2}}. \end{aligned}$$

Then $\partial\Omega/\partial z = -u + v \geqslant -u + v_0$ where v_0 is v evaluated at $z = 0$. Then

$$v_0^2 - u^2 = \frac{9n^3}{4(n-1)}\left((n-m-1)(nm-3n+4)x^2 - \left(2ny(n-m-1)\right)x + y^2(n^2-5n+4)\right).$$

The coefficient of the x^2 term is clearly positive for $m \geqslant 3$. The discriminant of $v_0^2 - u^2$ (omitting the factor $\frac{9n^3}{4(n-1)}$) is

$$-4y^2(n-2)^2(nm-4n+4)(n-m-1) < 0$$

for $m \geqslant 4$. Hence $\partial\Omega/\partial z > 0$ for all $m \geqslant 4$.

Putting these facts together we have that

(i) $\Omega(0) \geqslant 0$
(ii) $\partial\Omega/\partial z$ is an increasing function
(iii) $\partial\Omega/\partial z > 0$ for all $m \geqslant 4$.

This implies $\Omega \geqslant 0$ for $m \geqslant 4$ in case (2) with $z > 0$.

For $m = 3$ the discriminant of $v_0^2 - u^2$ may be positive. Solving $\partial\Omega/\partial z$ at $m = 3$ for z yields

$$z_0 = \frac{(2x-y)(n-4)(ny-2x-y)}{(n-2)^2}.$$

Note that if $z_0 < 0$ then $\Omega \geqslant 0$ since $\Omega(0) \geqslant 0$ and the derivative is increasing for $z \geqslant 0$. Letting $x = y + t$ for $t > 0$ we have that $z_0 \geqslant 0$ if $(n-3)y \geqslant 2t$. Substituting $m = 3, z = z_0$ and $x = y + t$ into Ω yields $z_1(y)z_2(y)$ where

$$\begin{aligned} z_1 &= (2n-5)(n-3)y^3 + 3t(2n-5)(n-4)y^2 + 3t^2(3n-8)(n-4)y \\ &\quad + 4t^3(n-3)(n-4) \text{ and} \\ z_2 &= (n-3)(14n^2-51n+36)y^3 + 3t(n-4)(14n^2-51n+36)y^2 \\ &\quad + 3t^2(13n^3+304n-120n^2-192)y + 4t^3(96n-33n^2+3n^3-64). \end{aligned}$$

Clearly z_1 is nonnegative for all $n \geqslant 4$. The coefficients of y^3 and y^2 in z_2 are nonnegative for $n \geqslant 4$. Note that the coefficient of y is negative for $n \in \{4, 5\}$ and the constant term is negative for $n \in \{4, 5, 6\}$, (and these coefficients are nonnegative for $n \geqslant 7$). But using the fact that $z_0 \geqslant 0$ if $(n-3)y \geqslant 2t$ we have that $z_2 \geqslant 0$ for $n \in \{4, 5, 6\}$. Hence we have that $\Omega(z_0) \geqslant 0$ when $m = 3$. Thus we have that $\Omega \geqslant 0$ for all $m \geqslant 3$ in case (2). The situation for $m = 2$ will be shown explicitly. Write $\Omega = A + B$ with A and B as before, and expand $H = (n-2)^2B^2 - (n-1)A^2$.

Make the substitution $x = y + t$ in H, to get a polynomial in t of degree 6 given by

$$\begin{aligned} f(t) &= a_1t^6 + a_2t^5 + a_3t^4 + a_4t^3 + a_5t^2 + a_6t + a_7 \text{ where} \\ a_1 &= 18n^2(n-3)^2(n-4) \geqslant 0 \text{ since } n \geqslant 4, \\ a_2 &= 72n^2y(n-3)^2(n-4) \geqslant 0 \text{ since } n \geqslant 4 \text{ and } y > 0, \\ a_3 &= 9n^2(n-3)^2(nz + 14ny^2 + 2z - 52y^2) \geqslant 0 \text{ since } n \geqslant 4, \\ a_4 &= 2n^2y(n-3)(9n^2 + 62n^2y^2 + 596y^2 + 9nz - 386ny^2 - 90z) \geqslant 0, \\ &\quad \text{since } 9n^2z > 90z \ \forall n \geqslant 4 \text{ and } 596y^2 + 62n^2y^2 > 386ny^2 \ \forall n \geqslant 4, \\ a_5 &= 3n^2(n-3)(24n^2y^4 + 4n^2y^2z + 20ny^2z + 188y^4 + 3z^2 - 136ny^4 - 68y^2), \\ a_6 &= 6n^2y(n-3)(n-2)(2ny^2 + nz + z - 6y^2)(2y^2 - z), \text{ and} \\ a_7 &= n^2(n-2)^2(2y^2-z)^2(ny^2 + nz - 3y^2) \geqslant 0 \text{ since } n \geqslant 4. \end{aligned}$$

Combining a_5, a_6 and a_7 into a quadratic in t and evaluating the discriminant of

$$g(t) = a_5t^2 + a_6t + a_7$$

yields

$$\begin{aligned} &-12n^4(n-2)^2(n-3)(2y^2-z)^2(12n^3y^6 + 272ny^6 + 16n^3y^4z + 24y^4nz \\ &\quad + 96y^4z + 23y^2n^2z^2 + n^3y^2z^2 + 3nz^3 - 100n^2y^6 - 240y^6 - 68n^2y^4 - 50ny^2z^2). \end{aligned}$$

Clearly this discriminant is non-positive if the final factor is nonnegative. This final factor is

$$4(3n-10)(n-3)(n-2)y^6 + 4z(n-2)(4n^2-9n-12)y^4 + nz^2(n-2)(n+25)y^2 + 3nz^3$$

which, as a polynomial in y, has nonnegative coefficients for all $n \geqslant 4$. Thus the discriminant of $g(t)$ is non-positive.

Since

$$g(0) = n^2(n-2)^2(2y^2 - z)^2(ny^2 + nz - 3y^2) \geqslant 0$$

we have that $g(t)$ is nonnegative for all t. Hence $f(t)$ is also nonnegative, since $a_1, a_2, a_3, a_4, \geqslant 0$ and $g(t) \geqslant 0$. Thus we have shown that $\Omega \geqslant 0$ in case (2) with $z > 0$ and $m = 2$.

Finally we dispose of the cases when $m = 0$ or $m = 1$. Firstly suppose $m = 0$. By the lemma and, in particular, the result which says that $w' < w$, we can reduce to the case where $x_1 = x_2 = \cdots = x_{n-2} = x \geqslant x_{n-1} \geqslant x_n > 0$. We let $x_{n-1} = y$ and $x_n = w$, where $y, w > 0$. Thus $\Omega = A + \frac{n-2}{\sqrt{n-1}}B^{\frac{3}{2}}$ where

$$A = n^2S_3 + \frac{3}{2}n^2z(y+w) - 3nS_1S_2 - 3nS_1z + 2S_1^3 \text{ and}$$
$$B = (nS_2 + nz - S_1^2)$$

and

$$S_j = (n-2)x^j + y^j + w^j, j = 1, 2, 3.$$

First assume $z = 0$, then

$$(n-2)^2B^3 - (n-1)A^2 = n^2(n-2)^2(x-y)^2(x-w)^2Q(x, y, w),$$

where $Q = (4n-12)x^2 + (3n-3)y^2 + (3n-3)w^2 + (12-4n)xy + (12-4n)xw - (2n+6)yw$.

The symmetric matrix associated to the quadratic form Q is

$$\begin{pmatrix} 4n-12 & 6-2n & 6-2n \\ 6-2n & 3n-3 & -3-n \\ 6-2n & -3-n & 3n-3 \end{pmatrix}$$

which has eigenvalues $0, 4n, 6(n-3)$, and so is positive semidefinite for all $n \geqslant 3$. Thus Q is nonnegative for all $n \geqslant 3$.

For $z > 0$

$$(n-2)^2B^3 - (n-1)A^2 = \frac{n^2}{4}(n-2)^2(2xy + 2xw - 2x^2 - 2y3w + z)^2(Q + 4nz).$$

Clearly $(n-2)^2B^3 - (n-1)A^2$ is nonnegative for all $n \geqslant 3$ since $z \geqslant 0$ and $Q \geqslant 0$ for all $n \geqslant 3$. Note that as before $\Omega \geqslant 0$ if $(n-2)^2B^3 - (n-1)A^2 \geqslant 0$, since $B \geqslant 0$. Notice this result also covers the case when $m = 1$ since then $x_n = w = 0$. This completes the proof of the main theorem. $\square$

Observation 1. The expression $\Phi = n^2s_3 - 3ns_1s_2 + 2s_1^3 + \frac{n-2}{\sqrt{n-1}}(ns_2 - s_1^2)^{\frac{3}{2}}$ is invariant under the translation

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \to (\lambda_1 + h, \lambda_2 + h, \ldots, \lambda_n + h)$$

for all $h \in \mathbb{R}$, where $s_j = \sum_{i=1}^n \lambda_i^j, j = 1, 2, 3$.

Observation 2. Equality is achieved in Φ for matrices of the form $diag(x, x, \ldots, x, 0)$.

Observation 3. In [3] the authors consider the NIEP from the viewpoint of the coefficients of the characteristic polynomial of a realizing matrix and an associated weighted digraph. In Theorem 3 part (c) they give a general necessary inequality related to $\Phi \geqslant 0$.

3. An application of the inequality

We now discuss perturbations on realizable lists preserving realizability. Guo [4], in Theorem 3.1 proves.

Theorem 4. *Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ be realizable by a nonnegative matrix, where λ_1 is the Perron root and λ_2 is real. Then, for all $t \geqslant 0$, the list $(\lambda_1 + t, \lambda_2 + \varepsilon t, \lambda_3, \ldots, \lambda_n)$ is also realizable for all $\varepsilon \in [-1, 1]$.*

Laffey [6], in Theorem 1.1 (see [5] for an alternative proof) proves an analogue of Guo's theorem in the non-real case by showing

Theorem 5. *Let $\sigma = (\lambda_1, \lambda_2, \overline{\lambda_2}, \lambda_4, \ldots, \lambda_n)$ be realizable by a nonnegative matrix, where λ_1 is the Perron root and λ_2 and $\overline{\lambda_2}$ are non-real complex conjugates. Then, for all $t \geqslant 0$, the list $(\lambda_1 + 2t, \lambda_2 - t, \overline{\lambda_2} - t, \lambda_4, \ldots, \lambda_n)$ is also realizable.*

Note that, for this result, the trace of a realizing matrix for the perturbed list is unchanged. Hence, in general, $c = 2$ is the smallest multiple of t for which the result can hold for $(\lambda_1 + ct, \lambda_2 - t, \overline{\lambda_2} - t, \lambda_4, \ldots, \lambda_n)$. In 2007, Guo and Guo [5], in Proposition 3.1 shows that

Theorem 6. *Let $\sigma = (\lambda_1, \lambda_2, \overline{\lambda_2}, \lambda_4, \ldots, \lambda_n)$ be the spectrum of a nonnegative matrix, where λ_1 is the Perron root and λ_2 and $\overline{\lambda_2}$ are non-real complex conjugates. Then, for all $t \geqslant 0$, the list $(\lambda_1 + 4t, \lambda_2 + t, \overline{\lambda_2} + t, \lambda_4, \ldots, \lambda_n)$ is also realizable.*

The authors pose the problem of finding the smallest c for the which the result holds with ct in place of $4t$. Note that, unlike the situation in Theorem 5 above, there is no obvious restriction on the minimum c here other than $c \geqslant 1$. The requirement $c \geqslant 1$ follows since if $0 < c < 1$ then $|\lambda_1 + ct| < |\lambda_2 + t|$ for all sufficiently large t, thus contradicting the Perron condition for realizable spectra. We now show, via a constructive method, that $c = 1$ is not sufficient even when $Re(\lambda_2) \geqslant 0$.

4. Example

Consider the realizable list $(\rho, \lambda, \overline{\lambda})$ where ρ is the Perron eigenvalue. Then clearly the list $(\rho + t, \lambda + t, \overline{\lambda} + t)$ is realizable, for all $t \geqslant 0$, by adding tI_3 to a realizing matrix for the original list of three numbers. Thus for $n = 3$ at least, $c = 1$ is sufficient for realizability. Next consider the realizable list of four numbers $(\rho, \lambda, \overline{\lambda}, \mu)$ where μ is a real number. By the result of Guo from Theorem 4 above we know that $(\rho + t, \lambda, \overline{\lambda}, \mu - t)$ is realizable for all $t \geqslant 0$, since $\mu \in \mathbb{R}$. Hence we can simply add tI_4 to a realizing matrix to get that the list $(\rho + 2t, \lambda + t, \overline{\lambda} + t, \mu)$ is realizable for all $t \geqslant 0$. Thus $c = 2$ is sufficient for the list to be realizable when $n = 4$. But can we do better than $c = 2$ when $n = 4$?

Consider the particular example of four numbers given by $(\rho, i, -i, 0)$, where $i = \sqrt{-1}$.
We know that from (2), in order to be realizable this list must satisfy $4s_2 - s_1^2 \geqslant 0$, and hence we must have that $\rho \geqslant \sqrt{8/3}$.

We will now consider the case where $\sigma = (\sqrt{8/3}, i, -i, 0)$. Then σ is realizable, for example by the 4×4 nonnegative matrix $A = \alpha I + C$ where $\alpha = \frac{\sqrt{8/3}}{4} = \frac{1}{\sqrt{6}}$ and C is the companion matrix of

$$
\begin{aligned}
f(x) &= \left(x - \left(\sqrt{8/3} - \frac{\sqrt{8/3}}{4}\right)\right)\left(\left(x + \frac{\sqrt{8/3}}{4}\right)^2 + 1\right)\left(x + \frac{\sqrt{8/3}}{4}\right) \\
&= x^4 - \frac{5\sqrt{6}}{9}x - \frac{7}{12},
\end{aligned}
$$

that is

$$A = \begin{pmatrix} \frac{1}{\sqrt{6}} & 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 1 \\ \frac{7}{12} & \frac{5\sqrt{6}}{9} & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

For a related discussion on the realizability of lists with negative real parts see [7].

Let $\sigma_c := (\sqrt{8/3} + ct, t + i, t - i, 0)$ where $t \geqslant 0$. We first show that if $c = 1$, there is a $t > 0$ for which σ is not realizable. A necessary condition for realizability is that $\Phi \geqslant 0$, i.e.

$$\Phi = 16s_3 - 12s_1s_2 + 2s_1^3 + \frac{2}{\sqrt{3}}\left(4s_2 - s_1^2\right)^{\frac{3}{2}} \geqslant 0,$$

where $s_j = \left(\sqrt{\frac{8}{3}} + ct\right)^j + (t+i)^j + (t-i)^j,\ j = 1, 2, 3$. Letting

$$\begin{aligned} &A = 16s_3 - 12s_1s_2 + 2s_1^3 = -6t^3 - 4\sqrt{6}t^2 - 8t + \frac{80}{3}\sqrt{6},\ \text{and} \\ &B = 4s_2 - s_1^2 = 3t^2 + \frac{4}{3}\sqrt{6}t,\ \text{we get that} \\ &\frac{4}{3}B^3 - A^2 = \left(36t^6 + 48\sqrt{6}t^5 + 128t^4 + \frac{512}{27}\sqrt{6}t^3\right) \\ &\qquad - \left(36t^6 + 48\sqrt{6}t^5 + 192t^4 - 256\sqrt{6}t^3 - 1216t^2 - \frac{1280}{3}\sqrt{6}t + \frac{12800}{3}\right) \\ &\qquad = -64t^4 + \frac{7424\sqrt{6}}{27}t^3 + 1216t^2 + \frac{1280\sqrt{6}}{3}t - \frac{12800}{3}. \end{aligned}$$

Thus for $c = 1$ the condition $\Phi \geqslant 0$ fails for $t > t_0$, where $t_0 \approx 12.15964317$ is the largest real root of

$$64t^4 - \frac{7424\sqrt{6}}{27}t^3 - 1216t^2 - \frac{1280\sqrt{6}}{3}t + \frac{12800}{3} = 0. \tag{4}$$

So we can conclude that the list $(\sqrt{8/3} + t, t + i, t - i, 0)$ is not realizable for all $t > t_0$. Hence $(\rho + t, \lambda + t, \bar{\lambda} + t, \mu)$ need not be realizable for all $t > 0$, whenever $(\rho, \lambda, \bar{\lambda}, \mu)$ is realizable.

We will now show that the list $\sigma_1 = (\sqrt{8/3} + t, t + i, t - i, 0)$ is realizable for all $0 \leqslant t \leqslant t_0$. Consider the matrix $hI_4 + A$ where A has the form

$$A = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \\ u & v & 0 & a \end{pmatrix}.$$

We will show that the list

$$\sigma - h := \left(\sqrt{8/3} + t - h, t + i - h, t - i - h, -h\right),$$

where the kth power sums of $\sigma - h$ are

$$s_k := \left(\sqrt{8/3} + t - h\right)^k + (t + i - h)^k + (t - i - h)^k + (-h)^k, \quad k = 1, 2, \ldots,$$

is realizable by A.

Note that for the list $\sigma - h$, the condition $\Phi \geqslant 0$ is satisfied for all $0 \leqslant t \leqslant t_0$ by (4) and Observation 1.

The characteristic polynomial of A is

$$f(x) = x^4 - 3ax^3 + 3a^2x^2 - (a^3 + v)x + av - u.$$

Note that A satisfies $3s_2 - s_1^2 = 0$. Let $h = \frac{\sqrt{6}}{6} + \frac{3t}{4} - \frac{1}{4}\sqrt{4\sqrt{6}t + 9t^2}$ which is, for $t > 0$, the smallest positive root of the equation $3s_2 - s_1^2 = 0$. Then $a = \frac{s_1}{3} = \left(\sqrt{\frac{8}{3}} + 3t - 4h\right)/3 = \frac{1}{3}\sqrt{4\sqrt{6}t + 9t^2} > 0$ for all $t > 0$. We find v by solving $tr(A^3) - s_3 = 0$, to get

$$v = \left(\frac{1}{24}t^{\frac{5}{2}} + \frac{\sqrt{6}}{54}t^{\frac{3}{2}}\right)\left(\sqrt{(4\sqrt{6} + 9t)}\right) + \frac{5}{9}\sqrt{6} - \frac{1}{6}t - \frac{1}{12}\sqrt{6}t^2 - \frac{1}{8}t^3$$

and note that $v = \frac{\Phi}{48}$ and so $v \geqslant 0$ for all $0 \leqslant t \leqslant t_0$, since $\Phi \geqslant 0$ for all $0 \leqslant t \leqslant t_0$. Finally we find u by solving $tr(A^4) - s_4 = 0$. Since $av - u = det(A) = \left(\sqrt{\frac{8}{3}} + t - h\right)(t + i - h)(t - i - h)(-h) < 0$ we find that $u \geqslant 0$ for all $t \leqslant t_0$ since $v \geqslant 0$ for all $t \leqslant t_0$, and $a \geqslant 0$. Thus we have shown that the list σ_1 is realizable for all $0 \leqslant t \leqslant t_0$ by the matrix $hI_4 + A$ where A is as above. We now ask the natural question: What is the smallest $c \geqslant 1$ for which σ_c is realizable for all $t > 0$? We first check the necessary inequality proven earlier. Recall, that this says

$$\Phi := n^2 s_3 - 3ns_1s_2 + 2s_1^3 + \frac{n-2}{\sqrt{n-1}}\left(ns_2 - s_1^2\right)^{\frac{3}{2}} \geqslant 0.$$

We want to find the smallest c for which $\Phi(t, c) \geqslant 0$ for all $t \geqslant 0$. Let

$$A = n^2 s_3 - 3ns_1s_2 + 2s_1^3$$

and

$$B = \frac{n-2}{\sqrt{n-1}}\left(ns_2 - s_1^2\right)^{\frac{3}{2}}.$$

Then substituting $n = 4$ and σ_c into $\overline{\Phi} = B^2 - A^2$ we get that

$$\overline{\Phi} = a_1c^4 + a_2c^3 + a_3c^2 + a_4c + a_5,$$

where

$$\begin{aligned}
a_1 &= 576t^6 - 1728t^4,\\
a_2 &= -1408t^6 + 1536\sqrt{6}t^5 + 5760t^4 - 4608\sqrt{6}t^3,\\
a_3 &= 1344t^6 - 2816\sqrt{6}t^5 + 1920t^4 + 11520\sqrt{6}t^3 - 27072t^2,\\
a_4 &= -768t^6 + 1792\sqrt{6}t^5 - 8192t^4 - 5632\sqrt{6}t^3 + 49920t^2 - 11520\sqrt{6}t,\\
a_5 &= 256t^6 - 512\sqrt{6}t^5 + 2048t^4 - \frac{4096\sqrt{6}}{9}t^3 - 19200t^2 + 12800\sqrt{6}t - 12800.
\end{aligned}$$

Recall that, $\overline{\Phi}$ and $\overline{\Phi}'$ have a common root where the discriminant of $\overline{\Phi}$ (as a polynomial in t) is zero. The discriminant of Φ, as a function of t, factors as

$$D_c = 2^{71}5^3(c-1)^2(3c^2+8)^2(3c^2-12c+4)^3(81c^4-360c^3+360c^2-480c+400)$$
$$(45c^4-72c^3+72c^2-96c+80)^3.$$

We know from earlier that $c = 1$ is not possible for realizability. The roots of $3c^2 - 12c + 4$ are approximately 0.3670068381 and 3.632993162. The only other real roots of D_c are $c_1 \approx 1.001936284$ and $c_2 \approx 3.552840578$, where $c_1 =$

$$\frac{10}{9} + \frac{2}{9}\sqrt{10+6(5)^{\frac{2}{3}}} - \frac{1}{9}\sqrt{80-24(5)^{\frac{2}{3}} + \left(10-6(5)^{\frac{2}{3}}+18(5)^{\frac{1}{3}}\right)\sqrt{10+6(5)^{\frac{2}{3}}}}$$

is the smallest positive root of the quartic equation

$$81c^4 - 360c^3 + 360c^2 - 480c + 400 = 0.$$

So c_1 is an algebraic number. Upon substituting c_1 into Φ we see that $\Phi \geqslant 0$ is satisfied for all positive t provided $c \geqslant c_1$. Thus a necessary condition for the realizability of σ_c is $c \geqslant c_1$. We will now show that σ_c is realizable for all $c \geqslant c_1$. By a well known theorem of Brauer, it suffices to show that σ_c is realizable for $c = c_1$. We now show that a realizing matrix M of the form $hI_4 + B$ where

$$B = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \\ u & v & q & a \end{pmatrix}$$

can be found for $c = c_1$.

The characteristic polynomial of B is

$$f(x) = x^4 - 3ax^3 + (3a^2 - q)x^2 + (2aq - v - a^3)x + va - u - a^2q.$$

We have

$$trB^k = \left(\sqrt{\frac{8}{3}} + c_1t - h\right)^k + (t+i-h)^k + (t-i-h)^k + (-h)^k, \quad k = 1, 2, \ldots.$$

We will break the problem into two cases, one for small t and one for large t. We do the large t case first. For B to have the desired spectrum $\left(\sqrt{\frac{8}{3}} + c_1t - h,\, t \pm i - h,\, -h\right)$, we must have that

$$f(x) = \left(x - \left(\sqrt{\frac{8}{3}} + c_1t - h\right)\right)\left((x-t+h)^2 + 1\right)(x+h).$$

We let h be a linear multiple of t. In particular, let $h = t/10000$. Solving $s_1 - trB = 0$ for a gives

$$a = \frac{2}{9}\sqrt{6} + \frac{1}{3}c_1t + \frac{4999}{7500}t.$$

Substituting a into $s_2 - trB^2$ and solving for q gives

$$q = \left(\frac{c_1^2}{3} - \frac{19999}{30000}c_1 + \frac{50009999}{150000000}\right)t^2 + \left(\frac{4}{9}\sqrt{6}c_1 - \frac{19999}{45000}\sqrt{6}\right)t - \frac{1}{9}.$$

Substituting a and q into $s_3 - trB^3$ for v gives

$$
\begin{aligned}
v = {} & \left(\frac{5}{27}c_1^3 - \frac{10001}{45000}c_1^2 - \frac{99960001}{900000000}c_1 + \frac{999849985001}{6750000000000}\right)t^3 \\
& + \left(\frac{10}{27}\sqrt{6}c_1^2 - \frac{10001}{33750}\sqrt{6}c_1 - \frac{99960001}{1350000000}\sqrt{6}\right)t^2 \\
& + \left(\frac{49}{27}c_1 - \frac{259999}{135000}\right)t + \frac{134}{24}\sqrt{6}.
\end{aligned}
$$

Finally, substituting a, q and v into $s_4 - trB^4$ and solving for u gives

$$
\begin{aligned}
u = {} & \left(\frac{2}{81}c_1^4 - \frac{4001}{162000}c_1^3 - \frac{199979999}{2700000000}c_1^2 + \frac{10000299940001}{81000000000000}c_1 - \frac{40003999699980001}{810000000000000000}\right)t^4 \\
& + \left(\frac{16}{243}\sqrt{6}c_1^3 - \frac{4001}{81000}\sqrt{6}c_1^2 - \frac{199979999}{2025000000}\sqrt{6}c_1 + \frac{10000299940001}{121500000000000}\sqrt{6}\right)t^3 \\
& + \left(\frac{50}{81}c_1^2 + \frac{19951}{810000}c_1 - \frac{5199480001}{8100000000}\right)t^2 \\
& + \left(\frac{344}{729}\sqrt{6}c_1 + \frac{379933}{3645000}\sqrt{6}\right)t + \frac{560}{729}.
\end{aligned}
$$

It can be easily verified that all these entries are nonnegative for $t > 50$. Thus this B realizes $(\sqrt{8/3} + c_1 t - h, t \pm i - h, -h)$. Hence σ_{c_1} is realized by $M = hI_4 + B$ in this case.

We now consider the case where $0 < t \leqslant 50$. This time we do not assign a value to h from the outset. As before we solve $s_1 - trB = 0$ for a, and $s_2 - trB^2$=0 for q using this value for a. These solutions are functions of t and h. We label them $a(t, h)$ and $q(t, h)$ for notational convenience. We now solve the quadratic expression $q(t, h) = 0$ for h, and take the smaller of these two roots to be h. So

$$
h = t\left(\frac{c_1 + 2}{4}\right) + \frac{\sqrt{6}}{6} - \frac{\sqrt{(9c_1^2 - 12c_1 + 12)t^2 + (12c_1 - 8)\sqrt{6}t}}{4}.
$$

Clearly h is real. Also $h \geqslant 0$ for all $t \leqslant 50$ since

$$
\begin{aligned}
& \left(t\left(\frac{c_1 + 2}{4}\right) + \frac{\sqrt{6}}{6}\right)^2 - \frac{(9c_1^2 - 12c_1 + 12)t^2 + (12c_1 - 8)\sqrt{6}t}{16} \\
& = -\frac{1}{2}(c_1 - 1)^2 t^2 + \left(\frac{2}{3}\sqrt{6} - \frac{2}{3}\sqrt{6}c_1\right)t + \frac{1}{6}
\end{aligned}
$$

is a quadratic with negative leading term and roots approximately equal to -1738.185709 and 51.16617929.

We let

$$
b = (9c^2 - 12c + 12)t^2 + (12c - 8)\sqrt{6}t.
$$

We now solve $s_1 - trB = 0$ for a, and substituting the value of h in, we get

$$
a = \frac{\sqrt{(9c^2 - 12c + 12)t^2 + (12c - 8)\sqrt{6}t}}{3} = \frac{\sqrt{b}}{3}.
$$

We next solve $s_3 - trB^3 = 0$ for v, using a and h as above, to get

$$
v = \frac{1}{72}(3c^2t^2 + 4\sqrt{6}ct + 20)(3ct - 6t + 2\sqrt{6}) + \frac{b^{\frac{3}{2}}}{216}.
$$

Then letting

$$x = \frac{1}{72}(3c^2t^2 + 4\sqrt{6}ct + 20)(3ct - 6t + 2\sqrt{6})$$

and

$$y = \frac{b^{\frac{3}{2}}}{216}$$

and expanding $y^2 - x^2$ as a polynomial in t, we have $6912(y^2 - x^2) = \overline{\Phi}$. Thus we get that v is nonnegative precisely when Φ is nonnegative as in the earlier case when $c = 1$. Thus we have shown that σ_{c_1} is realizable by $M = hI_4 + B$ (with $q = 0$) for all $t \leqslant 50$.

In summary, we have shown that the list of four numbers given by $(\sqrt{8/3} + ct, t + i, t - i, 0)$ is realizable for all $t \geqslant 0$ when $c \geqslant c_1$.

5. Convexity of realizable sets

The results in this discussion show that the sum of two realizable lists need not be realizable, so the set of realizable lists in the NIEP do not form a convex set. To see this we note that the lists $\sigma = \left(\sqrt{\frac{8}{3}}, i, -i, 0\right)$ and $\tau = (1, 1, 1, 0)$ are both realizable. The first list is realizable by $\alpha I_4 + C$ where C is the companion matrix of trace zero given at the start of section 4 and τ is realizable by $I_3 \oplus 0$. Then from earlier we know that $\sigma + t(1, 1, 1, 0)$ is not realizable for $t = 15$, for example. Thus the convex combination $\frac{1}{16}\sigma + \frac{15}{16}\tau$ is not realizable. For a discussion on convex combinations of symmetrically realizable lists see [9].

6. Conclusion

In this paper we offer a new inequality of a JLL type for nonnegative matrices previously noted for $n = 4$ only. We use this inequality to discuss perturbation results on realizable lists. In particular we show that if $(\lambda_1, \lambda_2, \overline{\lambda_2}, \ldots, \lambda_n)$ is realizable, where λ_1 is the Perron root and λ_2 is non-real, then the list $(\lambda_1 + t, \lambda_2 + t, \overline{\lambda_2} + t, \ldots, \lambda_n)$ need not be realizable for all $t > 0$.

References

[1] R. Loewy, D. London, A note on the inverse eigenvalue problem for nonnegative matrices, Linear and Multilinear Algebra 6 (1978) 83–90.
[2] C.R. Johnson, Row stochastic matrices similar to doubly stochastic matrices, Linear and Multilinear Algebra 10 (1981) 113–120.
[3] J. Torre-Mayo, M.R. Abril-Raymundo, E. Alarcia-Estévez, C. Marijuán, M. Pisonero, The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs, Linear Algebra Appl. 426 (2007) 729–773.
[4] W. Guo, Eigenvalues of nonnegative matrices, Linear Algebra Appl. 206 (1997) 261–270.
[5] S. Guo, W. Guo, Perturbing non-real eigenvalues of non-negative real matrices, Linear Algebra Appl. 426 (2007) 199–203.
[6] T.J. Laffey, Perturbing non-real eigenvalues of non-negative real matrices, Electron. J. Linear Algebra 12 (2005) 73–76.
[7] T.J. Laffey, H. Smigoc, Nonnegative realization of spectra having negative real parts, Linear Algebra Appl. 416 (2006) 148–159.
[8] Meehan, E. Mary, Some results on matrix spectra, Ph.D. Thesis, National University of Ireland, Dublin, 1998.
[9] C. Knudsen, J.J. McDonald, A note on the convexity of the realizable set of eigenvalues for nonnegative symmetric matrices, Electron. J. Linear Algebra 8 (2001) 110–114.